



PERGAMON Computers and Mathematics with Applications 46 (2003) 681–684

An International Journal
**computers &
mathematics**
with applications

www.elsevier.com/locate/camwa

A Multiunit Generalization of a First Price Auction

YONG-HONG LONG

Information School, Renmin University of China, P.R. China
and
Institute for Advanced Study, Wuhan University, P.R. China*(Received May 2001; revised and accepted January 2003)*

Abstract—This paper studies equilibrium bidding in a first price multiunit auction, in which all agents have constant elasticity Von Neumann-Morgenstern utility functions. © 2003 Elsevier Ltd. All rights reserved.

Keywords—Auction, Equilibrium bid function, Constant relative risk averse.

1. INTRODUCTION

Vickrey [1] formulated a Nash equilibrium model of bidding by risk neutral economic agents in single unit auctions. Cox, Roberson and Smith [2] and Cox, Smith and Walker [3] studied a first price single unit auction between agents with constant relative risk aversion. When individual values for the auctioned object are assumed to be drawn from a uniform distribution, the equilibrium bid function can be expressed explicitly, and the expression shows that the equilibrium bid is independent of the distribution of the random parameter r in bidders' utility function $u(y) = y^r$. Thus, every bidder i can submit a bid the same as he submits in the case in which all his rivals have the same parameter r_i with himself. However, Long [4] gave an example to show that it is not the case for the multiunit auctions. In the present paper, we show that the equilibrium bids in multiunit auctions depend only on the expected value of r , when individual values are distributed uniformly. So this paper solves the equilibrium bid function for the multiunit in two steps. In the first step, we obtain the equilibrium bid function in the case in which all bidders are identical constant relative risk averse, and thus, in the case of random parameter with known expected value of r , we can obtain the equilibrium bid function for the parameter which is equal to the expected value. In the second step, we obtain that inverses of equilibrium bid functions are linear on r , and then, from the equilibrium bid function with random parameter which is equal to the expected value of r , we can compute the equilibrium bid with random parameter r . Harris and Raviv [5] consider a multiple unit auction in the case in which all agents have identical linear (risk neutral) and concave (risk averse) utility function, and obtain expression of risk neutral equilibrium bid function. Our result in this paper is the generalization.

2. MODEL

Let $Q \geq 1$ unit(s) of a homogeneous good be offered in perfectly inelastic supply to $N > Q$ bidders. Each bidder submits a bid for single unit with the understanding that each of the Q highest bidders will be awarded a unit of the good at a price equal to his bid; i.e., the institution is a discriminative sealed-bid auction. Let v_i be the monetary value of a unit of the good to bidder i , where $i = 1, 2, \dots, N$. Bidders are assumed to know their own v_i , but to know only the probability distribution with cdf $H(\cdot)$ on $[0, \bar{v}]$, from which their rivals' values are independently drawn. $H(\cdot)$ is assumed to have a continuous density function $h(\cdot)$ that is positive on $(0, \bar{v})$.

Suppose that all bidders are constant relative risk averse, i.e., each bidder i has a utility function as form $u_i(y) = y^{r_i}$, where r_i is a random variable with probability distribution $\Phi(\cdot)$ on $(0, \bar{r})$, where $\bar{r} \geq 1$. Each bidder is assumed to know his or her own r_i but to know only that r_j of his or her rivals' j is drawn from the probability distribution $\Phi(\cdot)$. Here, Φ may not have density function; it can have a mass of probability of 1. Therefore, included as special cases are the models where all bidders are risk-neutral or all bidders are identical constant relative risk-averse.

In this paper, we consider the symmetric equilibrium in differentiable strategies. Suppose that each bidder i expects each of his or her rivals to bid according to the differentiable bid function

$$b_i = b(v_i, r_i), \quad (1)$$

let $b(v, r)$ be strictly increasing in v and have the property that $b(0, r) = 0$ for all $r \in [0, \bar{r}]$, and denote by $\pi(b, r)$ the v -inverse of bid function $b(v, r)$. The probability that bidder i will bid less than or equal to some amount b in the range of (1) is

$$F(b) = \int_{[0, \bar{r}]} H(\pi(b, r)) d\Phi(r). \quad (2)$$

Hence, the probability that a bid b by i will win is the probability

$$G(b) = \binom{N-1}{Q-1} \int_0^b F(x)^{N-Q-1} [1 - F(x)]^{Q-1} dF(x). \quad (3)$$

From the result of [4] for a more general model, a differentiable function $b(v, r)$ is an equilibrium bid function if and only if its v -inverse $\pi(b, r)$ satisfies

$$\frac{\pi(b, r) - b}{r} = \frac{G(b)}{G'(b)}. \quad (4)$$

3. EQUILIBRIUM BIDS OF IDENTICAL RISK PREFERENCE MODELS

In this section, we consider the model, in which all bidders are identical constant relative risk averse. That is, all bidders have the same utility function $u(y) = y_r$, where r is the common constant relative risk parameter. Denote by $b_r(v)$ the equilibrium bid function, $\pi_r(b)$ the inverse of $b_r(v)$. Hence, from (2), we have

$$F(b) = H(\pi_r(b)). \quad (5)$$

Let

$$G_H(x) = \binom{N-1}{Q-1} \int_0^x H(v)^{N-Q-1} [1 - H(v)]^{Q-1} dH(v). \quad (6)$$

In fact, $G_H(\cdot)$ is the probability distribution of $(N - Q)^{\text{th}}$ -order statistic for a sample of size $N - 1$ from the distribution $H(\cdot)$. Equation (3) can be rewritten as follows:

$$G(b) = G_H(\pi_r(b)) = \binom{N-1}{Q-1} \int_0^{\pi_r(b)} H(v)^{N-Q-1} [1 - H(v)]^{Q-1} dH(v). \quad (7)$$

Thus, equation (4) can be rewritten to be

$$\frac{\pi_r(b) - b}{r} = \frac{G_H(\pi_r(b))}{G'_H(\pi_r(b)) \cdot \pi'_r(b)} \quad (8)$$

and the equilibrium bid function $b_r(v)$ satisfies

$$\frac{v - b_r(v)}{r} = \frac{G_H(v)}{G'_H(v)} \cdot b'_r(v). \quad (9)$$

Solving equation (9), we obtain

$$b_r(v) = \frac{1}{G_H^{1/r}(v)} \int_0^v x dG_H^{1/r}(x). \quad (10)$$

As a special case, when $r = 1$,

$$b_1(v) = \frac{1}{G_H(v)} \int_0^v x dG_H(x) \quad (11)$$

is the risk neutral equilibrium bid function which has been obtained by [5].

If $h(\cdot)$ is the constant density $(\bar{v})^{-1}$, then, $b_r(v)$ follows (10) and

$$G_H(x) = \binom{N-1}{Q-1} \int_0^{x/\bar{v}} Y^{N-Q-1} (1-Y)^{Q-1} dY. \quad (12)$$

Furthermore, if $Q = 1$, we have a very simple expression [2]

$$b_r(v) = \frac{n-1}{n-1+r} v. \quad (13)$$

4. EQUILIBRIUM BIDS OF NOT IDENTICAL RISK PREFERENCE

In this section, we concentrate on solving equation (4) in the case that individual values for the auctioned object(s) were assumed to be drawn from a uniform distribution, and bidders may be different constant relative risk averse as mentioned in Section 2.

As $h(\cdot)$ is the constant density $(\bar{v})^{-1}$, from equation (2), for the v -inverse $\pi(b, r)$ of any bid function $b(v, r)$,

$$F(b) = \frac{1}{\bar{v}} \int_{(0, \bar{r})} \pi(b, r) d\Phi(r) = \frac{1}{\bar{v}} E[\pi(b, r)]. \quad (14)$$

By equation (4), if $\pi(b, r)$ is an equilibrium bid function, then

$$\pi(b, r) = \frac{\pi(b, r') - b}{r'} \cdot r + b, \quad \text{for any } r, r' \in (0, \bar{r}). \quad (15)$$

Particularly, take $r' = E_r$, where E_r is the expected value of r , and equation (15) concludes that

$$\pi(b, r) = \frac{\pi(b, E_r) - b}{E_r} \cdot r + b. \quad (16)$$

Thus, by equation (14), we have

$$F(b) = \frac{1}{\bar{v}} \pi(b, E_r) = H(\pi(b, E_r)). \quad (17)$$

To solve equation (4), by equation, it sufficient to obtain $\pi(b, E_r)$. From equation (4), $\pi(b, E_r)$ satisfies

$$\frac{\pi(b, E_r) - b}{E_r} = \frac{G(b)}{G'(b)}, \quad (18)$$

where

$$G(b) = \binom{N-1}{Q-1} \int_0^b F(x)^{N-Q-1} [1 - F(x)]^{Q-1} dF(x). \quad (19)$$

By equation (17),

$$G(b) = G_H(\pi(b, E_r)) = \binom{N-1}{Q-1} \int_0^{\pi(b, E_r)} H(v)^{N-Q-1} [1 - H(v)]^{Q-1} dH(v), \quad (20)$$

where $H(\cdot)$ is the distribution function of the uniform distribution on $(0, \bar{v})$ with constant density $h(\cdot) = (\bar{v})^{-1}$.

Hence, it is completely similar to solve equations (7)–(9), and obtain the solution $b_r(v)$ which satisfies equation (10). We can solve equations (18), (20) to obtain that

$$b(v, E_r) = \frac{1}{G_H^{1/E_r}(v)} \int_0^v x dG_H^{1/E_r}(x), \quad (21)$$

where

$$G_H(x) = \binom{N-1}{Q-1} \int_0^{x/\bar{v}} Y^{N-Q-1} (1 - Y)^{Q-1} dY. \quad (22)$$

So, the solution $\pi(b, E_r)$ of equation (18) is the v -inverse of $b(v, E_r)$ defined by equation (21). This is an interesting result: one bidder, whose risk parameter is equal to the expected value of risk parameter, just acts the same as he or she does in the case of identical risk parameter. Finally, the solution of equation (4) is given by equations (21) and (16).

REFERENCES

1. W. Vickrey, Counterspeculation, auctions, and competitive sealed tenders, *Journal of Finance* **16**, 8–37, (March 1961).
2. J.C. Cox, B. Roberson and V.L. Smith, Theory and behavior of single unit auctions, In *Research in Experimental Economics, Volume 2*, (Edited by V.L. Smith), JAI Press, Greenwich, (1982).
3. J.C. Cox, V.L. Smith and J.M. Walker, Expected revenue in discriminative and uniform price sealed-bid auctions, In *Research in Experimental Economics, Volume 3*, (Edited by V.L. Smith), JAI Press, Greenwich, (1984).
4. Y. Long, Model of Q -unit discriminative auctions and the effect of risk preference on the equilibrium bids, Research Paper, Information School, Renmin University.
5. M. Harris and A. Raviv, Allocation mechanisms and the design of auctions, *Econometrica* **49**, 1477–1499, (November 1981).